Surface Water Waves
Superlensing and its Applications in Tsunami Defences

Alexander James Daniel
(4165737)

The University of Nottingham

16th May 2014
Abstract

The ability of modern, man-made materials, to refract light in unusual and unnatural ways is well known to science and the area of much research. Using a lattice of copper rods this phenomenon will be studied in the case of waves on the surface of water. By studying the propagation of waves it is possible to create the superlensing effect and experiment with its practical implications such as deflection of tsunamis from inhabited areas.
Surface Waves

Introduction

From the media we regularly hear how being able to alter the paths of electromagnetic waves through the use of new materials could lead to seemingly impossible feats. These include cloaking devices and perfect optical lenses free from chromatic aberrations. Using metamaterials they would enable light of visible wavelengths to be deflected around an object and continue its path on the far side as though the object were not there. This phenomenon has been achieved at microwave frequencies\textsuperscript{1,2} however, achieving this effect at optical frequencies involves scaling the structures inside the metamaterial down to a degree that is not possible with current technology. These technologies work by making materials exhibit a negative refractive index and thus allow the material to refract the wave in a way that is not seen in nature. The mechanism by which this negative refractive index is achieved is plasmon resonance in metamaterials\textsuperscript{2}. If a material could be created that exhibits a negative refractive index for waves propagating in water on large scales then it could potentially be used to deflect tsunamis from inhabited areas. In this case the negative refractive index would be achieved by Bragg scattering in a photonic crystal\textsuperscript{3-6} on a large scale.

By using a small bench top ripple tank we will study propagation of water waves with the goal of observing negative refraction leading to the superlensing effect\textsuperscript{4}. We will also find out the effects depth of water has on superlensing and if it is a feasible method for mitigating the effects of tsunamis. An understanding of the difference between surface water waves and deep water waves and the dominating factors in both of these must first be gained. This will make it possible for us to say if our observations would be the same in the ocean or if the effects we observe are limited to the ripple tank in the controlled conditions of the lab.
Surface Waves

Theory

It is first important to understand the difference between deep water waves and shallow water waves. The boundary between deep water and shallow water is not clear cut, nor is it at an absolute depth. It is instead defined by the ratio between the depth of the water and the wavelength of the waves propagating through it. If the depth in the water is much less than $\lambda/2$ the wave is a shallow water wave whereas if the depth is much greater than $\lambda/2$ the wave is a deep water wave. The particles in shallow water waves move in ovular shaped orbits whereas the particles in deep water waves move with approximately circular motion. In the region in between the two types of oscillatory motion the particles travel in ovals that become flatter as the distance below the water increases. This means that as long as we use wavelengths of the correct order we can generate both surface and deep water waves in the same ratio as they would occur in the sea. The ripple tank used is capable of generating waves with water depths between 2mm and 8mm therefore to have both surface and deep water waves the wavelengths must be of the order 5mm.

The speed a wave propagates through a liquid is governed by equation \(1\).

$$C = \sqrt{\frac{g}{k}} \tanh(kh) \left(1 + \frac{\gamma k^2}{\rho g}\right)$$

Equation 1

In this equation \(c\) is the wave speed, \(g\) is the force due to gravity, \(k\) is the wavenumber, \(h\) is the depth of the water, \(\gamma\) is the surface tension of the liquid and \(\rho\) is the density the fluid\(^7\). From this we can see that the speed will depend on both depth and wavenumber. The limits of \(\tanh\) are, \(\tanh(x) \approx 1\) for large values of \(x\) and \(\tanh(x) \approx x\) for small values of \(x\). This means that for a given wavenumber, if the surface tension term is excluded for shallow depths i.e. \(kh\) is small equation \(2\) holds and for deep depths equation \(3\) holds.

$$c \approx \sqrt{gh}$$

Equation 2

$$c \approx \sqrt{\frac{g}{k}}$$

Equation 3

This means that the speed of the wave is only dependent on wavenumber for deep water waves. The dependence of wave speed on wavenumber is called dispersion, therefore we can say the shallow water waves do not show dispersion but the deep water waves do show dispersion. If the capillary effect is included i.e. we do not ignore surface tension, then both shallow and deep water waves
Surface Waves

show dispersion because they both have a dependence on wavenumber as they are approximated by equations {4} and {5} respectively.

\[ c \approx \sqrt{gh + \frac{\gamma k^2 h}{\rho}} \]  \hspace{1cm} \text{Equation 4}

\[ c \approx \sqrt{\frac{g}{k} + \frac{\gamma k}{\rho}} \]  \hspace{1cm} \text{Equation 5}

These waves are both capillary and gravity waves. The difference between these two types of waves is the restoring force. For capillary waves, the restoring force is the surface tension of the water at the boundary between the water and the air whereas for gravity waves the restoring force is gravity acting on them. These two types of waves are both included in equation {1}.

Equation {1} can be plotted in three dimensions or as a series of curves to see how both wavenumber and depth affect the speed of the waves. These are shown in figures 1a and 1b.

![Figure 1 showing wave speeds at different depths and wavenumbers.](image)

It can be seen that for shallower depths there is no local minimum. This is because as the depth reduces, the capillary waves dominate over gravity waves because the proportion of the restoring force provided by surface tension is much greater than that provided by gravity.

The phenomenon of negative refraction is when instead of a wave being refracted by the angle expected by Snell’s law at the boundary between two mediums of different refractive indexes, it is refracted by the negative of this angle, this appears as though it has been reflected along the normal. The effect of negative refraction can be seen in figure 2.

![Figure 2](image)
Materials that allow this to occur are often called metamaterials or left-handed. By using these materials it is possible to focus waves as though they have gone through a lens, this process is called superlensing, it is possible to form a real image on the far side of a material[3][4]. For electromagnetic waves a negative refractive index metamaterial may be a photonic crystal, this is a structure that affects photons in the same way that an ionic lattice affects electrons. Properties like this can lead to photonic band structures and band gaps. In the photonic bands, the phase and group velocity of the propagating waves are in different directions, this can lead to the crystal being a metamaterial and exhibiting a negative refractive index. The material has a negative refractive index because it exhibits both a negative permittivity and permeability. The refractive index of a material is dictated by equation \[6\]

\[ n = \sqrt{\varepsilon \mu} \quad \text{Equation 6} \]

Usually the positive solution to this equation is assumed however some materials generate the negative solution thus leading to a negative refractive index. The photonic properties of the material are due to its lattice constant being smaller then the wavelength of the wave incident on it[6]. We will be trying to recreate an analogy of a photonic crystal in water waves.
A superlens is an optical device that uses negative refraction to go beyond the diffraction limit of ordinary lenses. An ideal superlens allows much finer detail to be resolved, hence its other name, a perfect lens. A slab of metamaterial with refractive index $n = -1$ relative to its surroundings would focus all rays of light to the same point, irrespective of their wavelength as seen in figure 3; this is not possible with regular lenses due to dispersion of rays within the lens\cite{8}.

If the refractive index of the metamaterial is not exactly negative one relative to its surroundings then the superlens will focus different wavelengths at different distances due to dispersion of the waves within the lens as can be seen in figure 4. This is therefore not a perfect lens. If the superlens is not perfect the pattern of water waves on the far side of the lens will be stretched horizontally as the wavelength is decreased.
Method

Experimental Setup

A bench top ripple tank was used to study the water waves. The apparatus consisted of a light emitting diode (LED) above the surface of the water. A plane of glass with edges around it to hold the water in, these edges were at 45° to the glass and covered in foam to reduce the reflections of the ripples off the edges. In addition to the foam, polyalkylene glycol was used as a surfactant to reduce the reflections by spreading it on the foam before water was added to the tank. Underneath this glass was a mirror pointing at a translucent Perspex screen. This allows the projection of any ripples to be easily seen with dark bands on the screen corresponding to peaks in the waveform and light bands corresponding to troughs. Finally there was a ripple generator, this was a small box that allowed various different pieces of apparatus to be clipped into its jaws and moved up and down in the water. The frequency and amplitude with which the objects oscillated could be controlled and a signal sent to the LED to make it flash at the same frequency as the oscillations or a small difference in frequency both higher and lower. The flashing light, or strobe, enabled the waves to appear stationary as the light was switched on at the same time in their cycle each time. If the frequency of the strobe was changed by a small amount from that of the wave driving frequency, the wave pattern would look like it moved in slow motion or was moving backwards. Using different attachments, or dippers, different waves could be produced e.g. point source or plane waves. To produce regular, consistent waves it was important that the water was clean, to aid this the tank was cleaned with solvents at the start of each day and fresh water was used.

Figure 5 showing the experimental setup.
The depth of water in the tank was measured by weighing it as it was added to the tank then, using the known surface area of the tank, the depth could easily be calculated. An approximation was made that the length of each side was its length half way up the 45° edge. Using this method relied on the tank being level though so before any water was added it was levelled using spirit levels.

This system magnified the projection of the waves on the screen therefore to measure the wavelength of any ripples this magnification had to be calculated. This was done by placing a coloured Perspex shape on the surface of the tank. With the LED switched on the shape was projected onto the screen and by comparing the measurements of the lengths of each edge with the length they appeared on the screen the magnification of the system was calculated.

**Dispersion**

We began by trying to find out if the surface wave approximation held for depths of approximately 5mm. To do this the tank was filled to 5±3mm and plane waves were generated using the correct attachment for the oscillator. The frequency of the wave was set using the dial on the oscillator and the LED was switched on. A ruler was placed in front of the screen and photographs were taken of the waves at eight different frequencies using a digital single lens reflex camera (DSLR). Using these photographs and the ruler the wavelength as it appeared on the screen could be calculated, this was then corrected to allow for the magnification of the system. The strobe could have been used to make the waves appear stationary rather than taking photographs however this made taking the measurements unpleasant and this method was less successful at low frequencies as the pause between flashes of light was too long. A graph of angular frequency against wavenumber was then plotted in Matlab. The gradient of this graph is the speed of the waves propagation across the surface of the water. This could then be compared to the surface wave approximation for wave speed, equation {2}.

The process of taking a photograph manually on the DSLR and comparing the patterns to the photographed ruler for each frequency to measure the wavelength was very time consuming. To speed up the process a Matlab script was written that took a black and white photograph using a USB camera and calculated the wavelength of the wave. This was done by dividing the image into 1D strips going down the image, each one of these 1D strips could be plotted as pixel intensity (how light or dark each pixel was) against number of pixels down the image. From this information the wavelength of the 1D slice could be calculated. By scanning across the entire photograph hundreds of these 1D slices...
could be averaged to produce the wavelength on the screen in units of pixels. This number needed to be converted into meters by calibrating the system using the ruler in one photograph and measuring the number of pixels per meter. Finally the value of the wavelength was corrected to allow for the magnification of the system. Using this script, with one button the wavelength of the water waves was calculated in a matter of seconds rather than minutes as it was before; this means many repeat readings can easily be taken. The value of wavelength the script was producing was checked against theory and the photographic method.

To investigate how the wave speed changes with the depth of the water the frequency of the oscillations was kept at a constant 7.5±0.1Hz while the depth of the water was changed. By using the script the velocity of the waves was measured and plotted against water depth.

Equation \{1\} is used to predict how the wave speed depends on wavenumber at different depths. It takes into effect dispersion and is not just a surface wave approximation. It was tested how the system behaves compared to this equation. To do this the wavelength and therefore velocity was measured at different frequencies and different depths of water. From this data a plot of wave speed against wavenumber was produced for each depth with the theoretical values also plotted.

**Superlensing**

To investigate superlensing a lattice of small copper rods was created\[^4\]. In addition to this a dipper to act as a point source was fabricated. Having tried different flange sizes ranging from a small blob of solder on the end of a copper rod to a 20mm flange turned on a lathe it was decided that the clearest, most intense waves, were produced by the 20mm flange. It was found that the best example of superlensing was produced using this dipper and the lattice. The effect became easier to observe if one of the rows was removed from the lattice however this also made the lensing effect decrease. It was not possible to take photographs of the superlensing effect with five rows in the lattice however photographs were taken when the lattice only had four rows.

Once the superlensing effect had been seen and photographed the implications that the depth of the water had on superlensing was investigated and photographed.

Figure 6 showing the shape of the dipper and the 20mm flange.
Surface Waves

Results

Having cleaned and levelled the tank, plane waves as shown in figure 7 could be seen. It was photographs such as figure 7 that were used to measure the wavelength before the use of the script. Using this method the angular frequency and wavenumber of plane waves in water of depth 0.005±0.003m were recorded and plotted in figure 8 to find the velocity of the waves.

The gradient of this plot was 0.19±0.02m⁻¹. Using equation (2) we calculate that the wave speed should be approximately 0.22±0.06m⁻¹. There was a y-intercept on this figure of 4±5s⁻¹.
Surface Waves

When the script was run it automatically took a photo, figure 9a, and analysed the intensity of each pixel in vertical slices and created a binary function as shown in red in figure 9b. From this binary function it was easy to calculate the wavelength in units of pixels then convert this into meters and compensate for the magnification of the system.

![Figure 9 showing how the computer processed the image.](image)

To confirm the script was working correctly we measured the wavelength of a 7.5±0.1Hz wave in water of depth 3.4±0.3mm using the ruler method and the script and compared both of these to the frequency expected by using equation \( \text{2} \). The theoretical wavelength was found to be 0.024±0.004m, the wavelength measured with a ruler was found to be 0.025±0.002m and finally the wavelength measured by the script was found to be 0.0216±0.0002m.

By measuring the wavelength at different depths it was possible to make a figure of wave speed against depth, figure 10. From this figure it was possible to say if the waves we were producing were showing dispersion and therefore if we were producing any deep water waves or just shallow waves. The theoretical values plotted on this figure are calculated using equation \( \text{2} \) and therefore do not have any dependence on wavenumber and can be considered to be none dispersive waves.
Having varied the depth at a set frequency, the frequency was varied at a series of depths and plots of wave speed against wavenumber were produced. This allowed the relationships between wavenumber and depth of water to be observed and see if these experimental findings match those predicted by equation \{1\}. Figures 11-14 were produced showing the experimental fits and the theoretical values predicted by equation \{1\}. These can be compared to figure 15; this figure shows theoretical values over the same range of wavenumbers and depths. It can be seen that our data roughly fits the theoretical model.
Surface Waves

Figure 11 showing wave speed against wavenumber at a depth of 2.7±0.2mm.

Figure 12 showing wave speed against wavenumber at a depth of 4.7±0.4mm.

Figure 13 showing wave speed against wavenumber at a depth of 6.6±0.5mm.

Figure 14 showing wave speed against wavenumber at a depth of 8.5±0.7mm.

Figure 15 showing the theoretical values for the range of depths and wavenumbers that were tested.
By using the 20mm flanged dipper we were able to produce very regular ripples from a point source as shown in figure 16. These ripples will make it clear when the superlensing effect has been achieved and therefore whether the copper lattice is exhibiting a negative refractive index.

A lattice with five rows of copper cylinders was placed into the tank and the frequency was adjusted until what looked like superlensing was observed. The effect was most visible at 8.4±0.1Hz however was very faint, this made it impossible to photograph. It was found that the best effect was achieved when the point source was very close to the lattice. Various different methods were employed to try and photograph the effect. The strobe was switched on to see if it could freeze the pattern allowing a longer exposure on the DSLR over multiple cycles however because of the relatively low frequency of the oscillation only minimal freezing was observed. This freezing was only observed on the waves before they entered the lattice, not after exiting, the strobe made the waves invisible after going through the lattice. We removed one row of cylinders from the lattice meaning the lattice now had four rows. When this row was removed the frequency at which superlensing was most apparent changed to 8.9±0.1Hz. The amplitude of the waves that had passed through the lattice was greatly increased by the removal of a row; this made it possible for the effect to be photographed, figure 17a. Another row was removed to leave three rows, once again the frequency which had the maximum amplitude increased to 9.1±0.1Hz, figure 17b. The angle that the wavefronts propagate increased as the number of rows was decreased.
Surface Waves

To see how the change in frequency was affecting the propagation angle the number of rows in the lattice was reset to four and the frequency was taken from 8.0±0.1Hz to 11.2±0.1Hz in 0.4±0.1Hz steps. The dispersion angle was observed to decrease as the frequency of the oscillations increased until 10.0±0.4Hz, figure 18. By imagining the wavefronts to have a ray associated with them as in classical optics it is possible to conclude that the system is producing a virtual image either inside or behind the lattice.

At this frequency the wavefronts coming from the lattice stopped resembling a point source and began to resemble plane waves i.e. there was very little curvature to each wavefront. As the frequency increased further the length of the wavefronts appeared to increase, figure 19.

It was decided to test how the depth of the water affected the superlensing so the oscillations were kept at a constant rate of 9.1±0.1Hz while the depth of water was changed. Photographs of the pattern were taken at depths of 3.2±0.2mm, 5.6±0.4mm and 8.0±0.6mm, these can be seen in figure 20. It can be seen that at 8.0±0.6mm the superlensing has stopped therefore superlensing is not possible at greater depths.

---

**Figure 18** showing superlensing through a four row lattice with a driving frequency of 10.0±0.1Hz.

**Figure 19** showing superlensing through a four row lattice with a driving frequency of 10.8±0.1Hz.

**Figure 20** showing waves of 9.1±0.1Hz propagating through a four row lattice with water depths of 3.2±0.2mm (left), 5.6±0.4mm (centre) and 8.0±0.1mm (right).
Discussion

The wave speed derived from figure 8 is within the 95% bound of the value anticipated by equation (2). The small deviation from the theoretical value could be due to the theoretical approximation not incorporating surface tension, however surface tension is a very small factor in this system. The deviation is probably due to an error in the measurement of the depth of water. The fact that the y-intercept is not at zero indicates that there is a systematic error in measurements. This could be caused by the oscillator being poorly calibrated or by the measurements of wavelength being consistently above the true value. One of the reasons for computerising the process of measuring wavelengths was to try and eliminate errors such as this. The value of wavelength measured by the script was compared to a measurement with the ruler and the value predicted by theory. It was decided that it was not time efficient to measure all the wavelengths manually so we would use the script.

In figure 10 it can be seen that the velocity of the wave changes with depth. For shallow depths the experimental values seem to follow the same curve as the theoretical ones with an offset on the x-axis. This offset could be due to a zero error in the depth measurement caused by, for example, the tanks drainage pipe being overfilled at the start of the experiment. As the depth of the water increases it can be seen that the theoretical curve and the experimental curves diverge. This is due to the fact that the theoretical curve does not allow for dispersion; it is only an approximation for small depths. This means that as the depth increases so does the effect of dispersion and therefore we can no longer use the approximation of equation (2) and should use equation (1).

In figures 11-14 the experimental points were fit to a linear model because they were only spanning a small range of wavenumbers therefore the function could be approximated as linear. It can be seen that the gradients of these lines are similar to the gradient of the theoretical values in all cases. It is also true that the gradients follow the same trend as depth increases i.e. they go from a positive gradient at the smallest depth to a negative gradient at the largest depth. This experimental data fits the theoretical model well therefore we can conclude that equation (1) is a good enough model for the waves in the ripple tank and that these waves are analogous to those in much deeper water or even the ocean.

When the point source was placed next to a lattice of five rows of copper rods and the source was oscillating at 8.4±0.1Hz superlensing was observed. This means that under those conditions the lattice had a negative refractive index. The intensity of the waves on the far side of the lattice was greatly reduced.
leading to photographs of superlensing being difficult to take. It was noticed that the strobe was not able to freeze the image of the refracted waves. This indicates that the waves propagating from the lattice are at a different frequency to those entering the lattice. This could be due to the interference they undergo whilst in the lattice causing them to exit with a different frequency. The removal of a row from the lattice increased the angle the wave fronts propagated through. This is expected because, in a classical optical analogy, the thickness of the prism has been reduced, in turn reducing the distance that the ray travels through the prism at the refracted angle. The dependence of frequency on propagation angle can be explained using classical optics also. As the frequency of the wave increases the angle it refracts through increases due to dispersion causing the virtual image to move further back thus decreasing the angle the waves propagate through. The effects both frequency and width of material have on the pattern are shown using the classical analogy of a converging lens followed by a diverging lens in figure 21. In this figure it can be seen that the higher frequency wave (blue) has a narrower dispersion angle and that the shorter distance between the lenses results in the wider dispersion angle. The fact that the pattern has a dependence on wavelength indicates that the material does not have a refractive index of exactly negative one relative to its surroundings. If it did the pattern would be the same for all wavelengths of incident waves.

When the superlensing effect was achieved it was clearest when the point source was very close to the lattice. This raises concerns for the practical applications for the structure as most sources of waves will be too far away to be considered point sources or will be generated as plane waves e.g. waves caused by the wind. In the case of tsunamis the point source (earthquake) will most likely be a long way from the coast therefore the wave must be assumed to be a plane wave. The fact that the lattice was in the water reduced the amplitude of the waves significantly and would reduce the impact of any waves even if the lattice did not act as a superlens.
Surface Waves

Conclusion

A ripple tank was set up to produce very regular plane waves. The wavelength of these waves was measured at a range of frequencies to estimate the velocity that the waves travelled across the surface of the water. It was found that the process of measuring the wavelength of waves using photographs of the waves next to a ruler was a slow process. To speed up data acquisition a script was written in Matlab to take a photograph and automatically calculate the wavelength in the photograph with an associated error. Using the mass of the water in the tank the depth was calculated; by varying the depth at constant frequency it was found that the waves that were being produced were dispersive and therefore were made up of both surface waves and deep water waves. The model to predict the motion of these waves would have to take into consideration the depth of the water they were propagating through. The frequency of the plane waves was next varied at a series of known depths. By plotting the wave speed against the wavenumber for each depth and comparing the resulting points to those predicted by theory using equation \{1\} it was possible to conclude that the waves produced in the tank were dispersive. This indicated that they behaved as expected and could be used as a scaled down model of the ocean.

A lattice of small copper cylinders was placed into the tank with a point source designed to produce the maximum amplitude of water wave. The lattice was chosen because of its potential to exhibit a negative refractive index when waves of a specific frequency band were incident upon it. If the lattice exhibited a negative refractive index then superlensing would occur. This would mean that an image of the point source would be formed on the far side of the lattice in some cases, in others the pattern of waves on the far side would be focused. It is this focusing property that was thought could have potential as a method of deflecting the energy of a tsunami away from inhabited areas by placing a scaled up version of the lattice into the sea. Having found the frequency that produced the most prominent focusing effect to be 8.4±0.1Hz when the lattice was five rows thick but a different frequency when the number of rows in the lattice decreased it was concluded that the degree of superlensing had a large dependence on lattice thickness. The frequency of the point source was varied at a constant depth of water. This meant that the waves that propagated away from the lattice began by looking as thought they had not been through the lattice. As the frequency was increased the superlensing effect began to become visible and the angle that the waves dispersed through after exiting the lattice decreased. The waves continued to become more focused as the frequency increased until a certain frequency, once again dependent on the number of rows in the lattice, where the waves propagating from the lattice began to resemble plane waves. The
frequency was increased further and the waves continued to become planar. The depth of the water around the lattice was changed whilst the frequency of the source was kept constant. The superlensing effect became less and less prominent as the depth increased. This, along with other results allowed us to conclude that superlensing structures in this form would not be an effective method of mitigating the effects of tsunamis.

References


[9] Figure generated using RayLab iPad by Kamyar Ghandi